Duality Relations and Exotic Orders in Electronic Ladder Systems

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We discuss duality relations in correlated electronic ladder systems to clarify mutual relations between various conventional and unconventional phases. For the generalized two-leg Hubbard ladder, we find two exact duality relations, and also one asymptotic relation which holds in the low-energy regime. These duality relations show that unconventional (exotic) density-wave orders such as staggered flux or circulating spin-current are directly mapped to conventional density-wave orders, which establishes the appearance of various exotic states with time-reversal and/or spin symmetry breaking. We also study duality relations in the SO(5) symmetry that was proposed to unify antiferromagnetism and d-wave superconductivity. We show that the same SO(5) symmetry also unifies circulating spin current order and s-wave superconductivity.

KEYWORDS: Electronic ladder, Hubbard model, duality relation, exotic order, SO(5) symmetry, staggered flux, spin circulating current.

1. Introduction

Unconventional density-wave orders such as d-density wave (dDW) (which is equivalently called as staggered flux or orbital antiferromagnets) and d-spin-density wave (dSDW) (which is circulating spin current) were first proposed in the context of excitonic insulators[1] and later discussed in high-T_c superconductors,[2–4] but the appearance of these orders was not established at that time. Recently, several experimental results have led to a resurgence of interest in the possibility of these exotic orders. A dDW state[5] was discussed to appear in the underdoped region of high-T_c superconductors, where a pseudogap was observed,[6] and also in the low-temperature phase of the quasi-two-dimensional organic conductor,[9] α-(BEDT-TTF)$_2$KHg(SCN)$_4$. Both a dSDW state[7] and a dDW state[8] were proposed as origins of hidden order in the heavy-fermion compounds URu_2Si_2 and UPt_3.

The appearance of unconventional (exotic) orders has been tested in microscopic models of correlated electrons. In particular, the generalized Hubbard model on the two-leg ladder has been attracting attention as a minimal model for showing the exotic orders.[10–15] As is well known, the standard two-leg Hubbard and t-J ladders without any inter-site interaction do not show any order.[11,14,16] Both strong-coupling and weak-coupling analyses however found that the generalized Hubbard ladder model with inter-site interactions exhibits various phases at half-filling.[10,12,13,15] There are at least eight phases,[13] i.e., charge-density-wave (CDW), dDW, p-density wave (pDW), f-density wave (fDW), $d^{(\prime)}$-Mott, and $s^{(\prime)}$-Mott phases. For less than half-filling, large-scale numerical calculations also reported CDW[17] and dDW[14] phases. Bosonaization and renormalization-group analysis revealed the appearance of quasi-long-range order of density waves.[15,18] Recently we found two exact duality relations between these various phases.[19] The relations show that unconventional density-wave orders such as staggered flux or circulating spin current are dual to conventional density-wave orders and there are direct one-to-one mappings between dual phases in the generalized Hubbard ladder systems.

The electronic ladder system also serves as a playground for theories of high-T_c superconductivity. The SO(5) theory, which was proposed to unify antiferromagnetism (AFM) and d-wave superconductivity (dSC) in terms of the SO(5) symmetry,[20–22] was also applied to the Hubbard ladder. Scalapino *et al.*[23] and later many groups[10–12,24] studied the Hubbard ladder with the SO(5) symmetry to capture some of the basic low-energy physics of the high-T_c cuprates. There, one grand order parameter field

$$(\sqrt{2}\mathrm{Re}\mathcal{O}_{d\mathrm{SC}}, \mathcal{N}_x, \mathcal{N}_y, \mathcal{N}_z, \sqrt{2}\mathrm{Im}\mathcal{O}_{d\mathrm{SC}}) \qquad (1)$$

behaves as a five-component vector, where $\mathcal{O}_{d\mathrm{SC}}$ denotes the pairing operator of dSC with $\mathrm{Re}\mathcal{O} = \frac{1}{2}(\mathcal{O}^\dagger + \mathcal{O})$, $\mathrm{Im}\mathcal{O} = \frac{1}{2i}(\mathcal{O}^\dagger - \mathcal{O})$, and the three elements $\mathcal{N}_\alpha$ with $\alpha = x, y, z$ are Cartesian components of the staggered magnetization.

In this paper we develop further the duality relations given in ref. 19 and also study the duality structure in the SO(5) symmetric Hubbard ladder. First, we discuss two exact duality relations in the generalized Hubbard ladder. One duality transformation relates conventional density waves to unconventional density (or current) waves and the other relates charge-density degrees of freedom to spin-current ones. These transformations also show duality relations between s- and d-wave superconductivity (sSC and dSC). Furthermore, we show another asymptotic duality relation, which appears only in the low-energy effective theory at half-filling. These duality relations among various density-wave phases are summarized in Fig. 1. The transformations give one-to-one parameter mappings between dual phases. If one finds an

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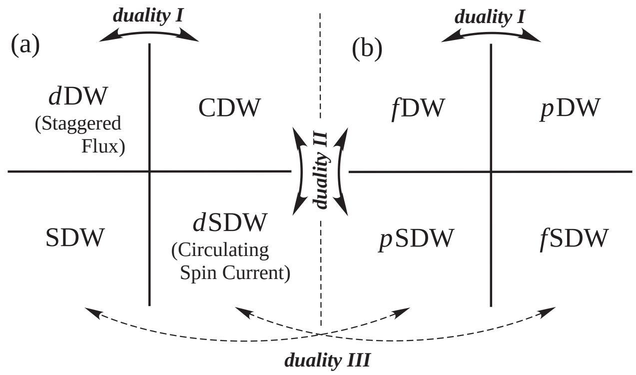

Fig. 1. Two exact duality relations (I and II) between four phases with (a) CDW, SDW, dDW, and dSDW orders, and (b) pDW, fDW, pSDW, and fSDW orders. There is also another duality relation (III), which appears only in the low-energy region at half-filling.

ordered phase in a certain parameter space, one can naturally conclude the appearance of dual ordered phases in dual parameter spaces. These relations help us to find various new exotic phases with spin and/or time-reversal symmetry breaking.

Next, we apply the duality relation to the SO(5) symmetry. We find that the SO(5) symmetry also unifies dSDW (circulating spin current) order and s-wave SC. Also, the SO(5) symmetric Hubbard ladder has a duality structure. In one parameter region, the original AFM-dSC SO(5) vectors represent dominant correlations, but in the other dual region the new dSDW and sSC vectors are dominant showing a crossover from dSDW to sSC upon doping.

This paper is structured as follows. The definitions of the Hamiltonian and operators are given in Sec. 2. The two exact duality relations I and II are described in Sec. 3. The duality structure in the generalized Hubbard ladder model is shown in Sec. 4. Section 5 contains a discussion on the duality in the SO(5) symmetry. Duality relations are reconsidered by using the Bosonization framework in Sec. 6. An application of the duality relations to results in the strong-coupling limit is discussed in Sec. 7. Section 8 contains discussions.

2. Definitions

2.1 Hamiltonian

We consider the generalized two-leg Hubbard ladder, which contains the on-site repulsion U, the intra-rung repulsion $V_\perp$, the intra-rung spin-exchange $J_\perp$ ($J_\perp^z$), and the intra-rung pair hopping $t_{\rm pair}$. The Hamiltonian is given by

$$H = H_0 + H_{\rm rung} - \mu \sum_{j,l,\sigma} n_{j,l,\sigma} \tag{2}$$

with

$$\begin{aligned} H_0 = -\sum_{j,\sigma} [\{ & t_\parallel (c^\dagger_{j,1,\sigma} c_{j+1,1,\sigma} + c^\dagger_{j,2,\sigma} c_{j+1,2,\sigma}) \\ & + t_\perp c^\dagger_{j,1,\sigma} c_{j,2,\sigma}\} + \text{H.c.}], \end{aligned} \tag{3a}$$

$$\begin{aligned} H_{\rm rung} = \sum_j \Bigg[& U \sum_l n_{j,l,\uparrow} n_{j,l,\downarrow} + V_\perp \sum_{\sigma,\sigma'} n_{j,1,\sigma} n_{j,2,\sigma'} \\ & + J_\perp (S^x_{j,1} S^x_{j,2} + S^y_{j,1} S^y_{j,2}) + J^z_\perp S^z_{j,1} S^z_{j,2} \\ & + t_{\rm pair} (c^\dagger_{j,1,\uparrow} c^\dagger_{j,1,\downarrow} c_{j,2,\downarrow} c_{j,2,\uparrow} + \text{H.c.}) \Bigg]. \end{aligned} \tag{3b}$$

Here $c^\dagger_{j,l,\sigma}$ ($c_{j,l,\sigma}$) ($l = 1, 2$ and $\sigma = \uparrow, \downarrow$) denotes an electron creation (annihilation) operator on the lth site of the jth rung, $n_{j,l,\sigma} = c^\dagger_{j,l,\sigma} c_{j,l,\sigma}$ and $\boldsymbol{S}_{j,l} = \frac{1}{2}\sum_{\sigma,\sigma'} c^\dagger_{j,l,\sigma} \boldsymbol{\sigma}_{\sigma\sigma'} c_{j,l,\sigma'}$ with the Pauli matrices σ^α ($\alpha = x, y, z$). In this paper we consider both spin isotropic ($J_\perp = J^z_\perp$) and anisotropic cases.

2.2 Order operators and typical states

Here, we list the definition of order operators used in the present work. We consider a charge-density-wave (CDW) operator (or s-density-wave operator)

$$\mathcal{O}_{\rm CDW}(j) = \frac{1}{2} \sum_\sigma (n_{j,1,\sigma} - n_{j,2,\sigma}), \tag{4}$$

a d-density-wave (dDW) operator

$$\mathcal{O}_{d{\rm DW}}(j) = \frac{i}{2} \sum_\sigma (c^\dagger_{j,1,\sigma} c_{j,2,\sigma} - \text{H.c.}), \tag{5}$$

a p-density-wave (pDW) operator (or staggered dimer operator)

$$\mathcal{O}_{p{\rm DW}}(j) = \frac{1}{4} \sum_\sigma [(c^\dagger_{j+1,1,\sigma} c_{j,1,\sigma} - c^\dagger_{j+1,2,\sigma} c_{j,2,\sigma}) + \text{H.c.}], \tag{6}$$

and an f-density-wave (fDW) operator (or diagonal current operator)

$$\mathcal{O}_{f{\rm DW}}(j) = \frac{i}{4} \sum_\sigma [(c^\dagger_{j+1,1,\sigma} c_{j,2,\sigma} - c^\dagger_{j+1,2,\sigma} c_{j,1,\sigma}) - \text{H.c.}]. \tag{7}$$

In the similar way, we consider operators in the spin sector. Inserting the Pauli matrix $\sigma^z_{\sigma\sigma}$ into the right-hand sides of eqs. (4)-(7), one defines a spin-density-wave (SDW) operator

$$\mathcal{O}_{\rm SDW}(j) = \frac{1}{2} \sum_\sigma \sigma^z_{\sigma\sigma} (n_{j,1,\sigma} - n_{j,2,\sigma}), \tag{8}$$

a d-spin-density-wave (dSDW) operator

$$\mathcal{O}_{d{\rm SDW}}(j) = \frac{i}{2} \sum_\sigma \sigma^z_{\sigma\sigma} (c^\dagger_{j,1,\sigma} c_{j,2,\sigma} - \text{H.c.}), \tag{9}$$

a p-spin-density wave (pSDW) operator $\mathcal{O}_{p{\rm SDW}}(j) = \frac{1}{4}\sum_\sigma \sigma^z_{\sigma\sigma}[(c^\dagger_{j+1,1,\sigma} c_{j,1,\sigma} - c^\dagger_{j+1,2,\sigma} c_{j,2,\sigma}) + \text{H.c.}]$, and an f-spin-density wave (fSDW) operator $\mathcal{O}_{f{\rm SDW}}(j) = \frac{i}{4}\sum_\sigma \sigma^z_{\sigma\sigma}[(c^\dagger_{j+1,1,\sigma} c_{j,2,\sigma} - c^\dagger_{j+1,2,\sigma} c_{j,1,\sigma}) + \text{H.c.}]$, respectively. Note that the CDW and pDW (SDW and pSDW) orders are kinds of density waves of charges (spins) while the dDW and fDW (dSDW and fSDW) orders finite local charge (spin) currents. Accurately speaking, dDW and fDW operators do not correspond to exact current

operators if there are pair-hopping terms ($t_{\text{pair}} \neq 0$) in the Hamiltonian, but they can detect time-reversal symmetry breaking, which is closely related to currents. Order parameters on the ladder are given by

$$O_A(q) = L^{-1} \sum_j \mathcal{O}_A(j) \exp(iqj),$$

where L is the number of rungs.

Moreover, we consider the d- and s-wave pairing operators,

$$\mathcal{O}_{d\text{SC}}(j) = \frac{1}{\sqrt{2}}(c_{j,1,\uparrow}c_{j,2,\downarrow} - c_{j,1,\downarrow}c_{j,2,\uparrow}), \tag{10}$$

$$\mathcal{O}_{s\text{SC}}(j) = \frac{1}{\sqrt{2}}(c_{j,1,\uparrow}c_{j,1,\downarrow} + c_{j,2,\uparrow}c_{j,2,\downarrow}), \tag{11}$$

which characterize the d- and s-wave spin-singlet superconductivity (dSC and sSC), respectively. Using these operators, the representatives of the d- and s-wave Mott-insulating (d- and s-Mott) states at half-filling are given by

$$|d\text{-Mott}\rangle = \prod_j \mathcal{O}^\dagger_{d\text{SC}}(j)|0\rangle, \qquad |s\text{-Mott}\rangle = \prod_j \mathcal{O}^\dagger_{s\text{SC}}(j)|0\rangle,$$

respectively, where $|0\rangle$ is the vacuum of the electron operators. We also consider the spin-triplet and spin-singlet pairing operators with odd parity given by

$$\mathcal{O}_{\text{t,o}}(j) = \frac{1}{\sqrt{2}}(c_{j,1,\uparrow}c_{j,2,\downarrow} + c_{j,1,\downarrow}c_{j,2,\uparrow}), \tag{12}$$

$$\mathcal{O}_{\text{s,o}}(j) = \frac{1}{\sqrt{2}}(c_{j,1,\uparrow}c_{j,1,\downarrow} - c_{j,2,\uparrow}c_{j,2,\downarrow}). \tag{13}$$

3. Exact duality transformations I and II

Here we describe the duality transformations on electron operators[19] so that this paper is self-contained. They are given by gauge transformations on the bonding and antibonding operators $d_{j,\pm,\sigma} = (c_{j,1,\sigma} \pm c_{j,2,\sigma})/\sqrt{2}$. Two duality transformations are presented in the following.

3.1 Duality relation I: density and current

Consider a gauge transformation of antibonding operators given by the unitary operator

$$U_I(\theta) = \prod_{j,\sigma} \exp(-i\theta d^\dagger_{j,-,\sigma} d_{j,-,\sigma}). \tag{14}$$

In the case of $\theta = \pi/2$, this operator gives the duality transformation I, $\tilde{d}_{j,\pm,\sigma} \equiv U_I(\pi/2) d_{j,\pm,\sigma} U_I(\pi/2)^{-1}$, which yields

$$\tilde{d}_{j,+,\sigma} = d_{j,+,\sigma}, \qquad \tilde{d}_{j,-,\sigma} = i d_{j,-,\sigma} \tag{15}$$

for $\sigma =\uparrow, \downarrow$. In terms of the electron operators $c_{j,l,\sigma}$, this transformation is written as

$$\begin{aligned} \tilde{c}_{j,1,\sigma} &= (e^{\pi i/4} c_{j,1,\sigma} + e^{-\pi i/4} c_{j,2,\sigma})/\sqrt{2}, \\ \tilde{c}_{j,2,\sigma} &= (e^{-\pi i/4} c_{j,1,\sigma} + e^{\pi i/4} c_{j,2,\sigma})/\sqrt{2}. \end{aligned} \tag{16}$$

Applying the transformation (15), we can obtain dual representations of operators straightforwardly as follows: The density-wave operators are transformed as

$$\begin{aligned} \tilde{\mathcal{O}}_{\text{CDW}} &= -\mathcal{O}_{d\text{DW}}, & \tilde{\mathcal{O}}_{d\text{DW}} &= \mathcal{O}_{\text{CDW}}, \\ \tilde{\mathcal{O}}_{\text{SDW}} &= -\mathcal{O}_{d\text{SDW}}, & \tilde{\mathcal{O}}_{d\text{SDW}} &= \mathcal{O}_{\text{SDW}}, \\ \tilde{\mathcal{O}}_{p\text{DW}} &= -\mathcal{O}_{f\text{DW}}, & \tilde{\mathcal{O}}_{f\text{DW}} &= \mathcal{O}_{p\text{DW}}, \\ \tilde{\mathcal{O}}_{p\text{SDW}} &= -\mathcal{O}_{f\text{SDW}}, & \tilde{\mathcal{O}}_{f\text{SDW}} &= \mathcal{O}_{p\text{SDW}}. \end{aligned} \tag{17}$$

Thus, current-order operators, such as dDW, dSDW, fDW, and fSDW operators, turn out to be dual to density-wave operators, CDW, SDW, pDW, and pSDW operators, respectively. It should be noted that the unitary operator $U_I(\theta)$ gives a continuous transformation between dual operators. For example, if one considers CDW and dDW operators, they are transformed as

$$U_I(\theta)\mathcal{O}_{\text{CDW}}U_I(\theta)^{-1} = \mathcal{O}_{\text{CDW}} \cos\theta - \mathcal{O}_{d\text{DW}} \sin\theta. \tag{18}$$

The s- and d-wave pairing operators are also related by the transformation (15) as

$$\tilde{\mathcal{O}}_{s\text{SC}} = \mathcal{O}_{d\text{SC}}. \tag{19}$$

Thus the d-wave SC phase is dual to the s-wave SC phase. By definitions, d-Mott and s-Mott phases are also dual to each other,

$$|\widetilde{d\text{-Mott}}\rangle = |s\text{-Mott}\rangle, \tag{20}$$

where we omitted a constant factor. On the other hand, the parity-odd pairing operators (12) and (13) are invariant under the transformation.

3.2 Duality relation II: density and spin current

Consider a gauge transformation given by the unitary operator

$$U_{II}(\theta) = \prod_j \exp\{-i\theta(d^\dagger_{j,+,\uparrow} d_{j,+,\uparrow} + d^\dagger_{j,-,\downarrow} d_{j,-,\downarrow})\}. \tag{21}$$

This unitary with $\theta = \pi/2$ gives the duality transformation II, $\bar{d}_{j,\pm,\sigma} \equiv U_{II}(\pi/2) d_{j,\pm,\sigma} U_{II}(\pi/2)^{-1}$, which yields

$$\begin{aligned} \bar{d}_{j,+,\uparrow} &= i d_{j,+,\uparrow}, & \bar{d}_{j,+,\downarrow} &= d_{j,+,\downarrow}, \\ \bar{d}_{j,-,\uparrow} &= d_{j,-,\uparrow}, & \bar{d}_{j,-,\downarrow} &= i d_{j,-,\downarrow}. \end{aligned} \tag{22}$$

In terms of the electron operators $c_{j,l,\sigma}$, this transformation is written as

$$\begin{aligned} \bar{c}_{j,1,\uparrow} &= (e^{\pi i/4} c_{j,1,\uparrow} + e^{3\pi i/4} c_{j,2,\uparrow})/\sqrt{2}, \\ \bar{c}_{j,2,\uparrow} &= (e^{3\pi i/4} c_{j,1,\uparrow} + e^{\pi i/4} c_{j,2,\uparrow})/\sqrt{2}, \\ \bar{c}_{j,1,\downarrow} &= (e^{\pi i/4} c_{j,1,\downarrow} + e^{-\pi i/4} c_{j,2,\downarrow})/\sqrt{2}, \\ \bar{c}_{j,2,\downarrow} &= (e^{-\pi i/4} c_{j,1,\downarrow} + e^{\pi i/4} c_{j,2,\downarrow})/\sqrt{2}. \end{aligned} \tag{23}$$

It is easily shown that the transformation (22) gives duality relations between density-waves operators as fol-

lows

$$\begin{aligned}
\bar{\mathcal{O}}_{\rm CDW} &= \mathcal{O}_{d{\rm SDW}}, & \bar{\mathcal{O}}_{d{\rm SDW}} &= -\mathcal{O}_{\rm CDW},\\
\bar{\mathcal{O}}_{d{\rm DW}} &= -\mathcal{O}_{\rm SDW}, & \bar{\mathcal{O}}_{\rm SDW} &= \mathcal{O}_{d{\rm DW}},\\
\bar{\mathcal{O}}_{p{\rm DW}} &= \mathcal{O}_{f{\rm SDW}}, & \bar{\mathcal{O}}_{f{\rm SDW}} &= -\mathcal{O}_{p{\rm DW}},\\
\bar{\mathcal{O}}_{f{\rm DW}} &= -\mathcal{O}_{p{\rm SDW}}, & \bar{\mathcal{O}}_{p{\rm SDW}} &= \mathcal{O}_{f{\rm DW}}.
\end{aligned} \tag{24}$$

Density waves of charges are transformed into spin currents while density waves of spins into charge currents. This transformation thus exchanges density and current as well as spin and charge degrees of freedom. Similarly to $U_I(\theta)$, the unitary operator $U_{II}(\theta)$ gives a continuous transformation between dual operators such as

$$U_{II}(\theta)\mathcal{O}_{\rm CDW}U_{II}(\theta)^{-1} = \mathcal{O}_{\rm CDW}\cos\theta + \mathcal{O}_{d{\rm SDW}}\sin\theta. \tag{25}$$

We note that under the transformation (22) the d-wave and s-wave pairing operators, $\mathcal{O}_{d{\rm SC}}$ and $\mathcal{O}_{s{\rm SC}}$, are invariant except for phase factors, and hence the d- and s-Mott states are also invariant. On the other hand, the triplet parity-odd paring operator $\mathcal{O}_{\rm t,o}$ is converted to the singlet parity-odd one $\mathcal{O}_{\rm s,o}$,

$$\bar{\mathcal{O}}_{\rm t,o} = \mathcal{O}_{\rm s,o}. \tag{26}$$

4. Duality relations in the electronic ladder

Here, we apply the duality transformations to the Hamiltonian (2). It can be shown that these transformations map the model onto the same Hubbard ladder with different coupling parameters. It is convenient to rewrite the Hamiltonian in terms of bonding and antibonding operators in the forms

$$\begin{aligned}
H_0 = -\sum_{j,\sigma}\Big[& t_\parallel \sum_{\lambda=\pm}(d^\dagger_{j,\lambda,\sigma}d_{j+1,\lambda,\sigma} + {\rm H.c.})\\
& + t_\perp(d^\dagger_{j,+,\sigma}d_{j,+,\sigma} - d^\dagger_{j,-,\sigma}d_{j,-,\sigma})\Big],
\end{aligned} \tag{27}$$

$$\begin{aligned}
H_{\rm rung} = \sum_j \Big[& A(d^\dagger_{j,+,\uparrow}d_{j,-,\uparrow}d^\dagger_{j,+,\downarrow}d_{j,-,\downarrow} + {\rm H.c.})\\
& + B(d^\dagger_{j,+,\uparrow}d_{j,-,\uparrow}d^\dagger_{j,-,\downarrow}d_{j,+,\downarrow} + {\rm H.c.}) + C\sum_\sigma n^{(d)}_{j,+,\sigma}n^{(d)}_{j,-,\sigma}\\
& + D\sum_{\lambda=\pm} n^{(d)}_{j,\lambda,\uparrow}n^{(d)}_{j,\lambda,\downarrow} + E\sum_{\lambda=\pm} n^{(d)}_{j,\lambda,\uparrow}n^{(d)}_{j,-\lambda,\downarrow}\Big],
\end{aligned} \tag{28}$$

where $n^{(d)}_{j,\lambda,\sigma} = d^\dagger_{j,\lambda,\sigma}d_{j,\lambda,\sigma}$, and $\lambda = +\ (-)$ represents the bonding (antibonding) orbital. The coupling constants in eq. (28) are given by

$$\begin{aligned}
A &= (U - V_\perp + t_{\rm pair})/2 + (2J_\perp + J^z_\perp)/8,\\
B &= (U - V_\perp - t_{\rm pair})/2 - (2J_\perp - J^z_\perp)/8,\\
C &= V_\perp + J^z_\perp/4,\\
D &= (U + V_\perp + t_{\rm pair})/2 - (2J_\perp + J^z_\perp)/8,\\
E &= (U + V_\perp - t_{\rm pair})/2 + (2J_\perp - J^z_\perp)/8.
\end{aligned}$$

Under the transformations (15) and (22), the total charge density, total magnetization, and kinetic energy term H_0 are invariant. Hence the chemical potential μ, magnetic field, and the parameters $t_\parallel$ and $t_\perp$ are unchanged through the mapping. On the other hand, the intra-rung coupling terms in $H_{\rm rung}$ are mixed up by the transformations. The parameter mappings are given in the following.

4.1 Duality relation I

It is easy to see in eq. (28) that the transformation (15) changes only the sign of the A-term, but keeps the rest of terms invariant. Hence, it gives a one-to-one parameter mapping

$$(A, B, C, D, E) \to (-A, B, C, D, E). \tag{29}$$

This leads to an exact duality relation in the parameter space of the system. The model with a parameter A is dual to the model with $-A$ and self-dual in the space $A = 0$, i.e.,

$$U - V_\perp + t_{\rm pair} + (2J_\perp + J^z_\perp)/4 = 0. \tag{30}$$

The coupling parameters are mapped as follows:

$$\begin{aligned}
\tilde{U} &= (U + V_\perp - t_{\rm pair})/2 - (2J_\perp + J^z_\perp)/8,\\
\tilde{V}_\perp &= (U + 3V_\perp + t_{\rm pair})/4 + (2J_\perp + J^z_\perp)/16,\\
\tilde{t}_{\rm pair} &= (-U + V_\perp + t_{\rm pair})/2 - (2J_\perp + J^z_\perp)/8,\\
\tilde{J}_\perp &= -U + V_\perp - t_{\rm pair} + (2J_\perp - J^z_\perp)/4,\\
\tilde{J}^z_\perp &= -U + V_\perp - t_{\rm pair} - (2J_\perp - 3J^z_\perp)/4.
\end{aligned} \tag{31}$$

Note that the spin isotropy ($J_\perp = J^z_\perp$) is conserved through this parameter mapping, i.e., $\tilde{J}_\perp = \tilde{J}^z_\perp$.

From the mapping (31) and the duality relation (17), one can conclude that if a density-wave order, e.g., CDW or SDW order, appears in a certain parameter region, a dual current order, i.e., dDW or dSDW order, respectively, exists in a corresponding dual parameter region. Because of this duality relation, all phase boundaries must be symmetric with respect to the self-dual space. Indeed, the transition line between the CDW and dDW phases derived for half-filling and in the weak- and strong-coupling limits[10,12,13] coincides with the self-dual line (30). We stress that our exact result holds in general cases, regardless of the coupling strength, filling, and system size.

4.2 Duality relation II

From eq. (28), one can see that the transformation (22) changes only the sign of the B-term and hence gives a one-to-one parameter mapping

$$(A, B, C, D, E) \to (A, -B, C, D, E). \tag{32}$$

Thus the present model has another duality: The model with a parameter B is dual to the model with $-B$ and self-dual in the space $B = 0$, i.e.,

$$U - V_\perp - t_{\rm pair} - (2J_\perp - J^z_\perp)/4 = 0. \tag{33}$$

The transformed coupling parameters are given by

$$\begin{aligned}
\bar{U} &= (U + V_\perp + t_{\mathrm{pair}})/2 + (2J_\perp - J_\perp^z)/8,\\
\bar{V}_\perp &= (U + 3V_\perp - t_{\mathrm{pair}})/4 - (2J_\perp - J_\perp^z)/16,\\
\bar{t}_{\mathrm{pair}} &= (U - V_\perp + t_{\mathrm{pair}})/2 - (2J_\perp - J_\perp^z)/8, \qquad (34)\\
\bar{J}_\perp &= U - V_\perp - t_{\mathrm{pair}} + (2J_\perp + J_\perp^z)/4,\\
\bar{J}_\perp^z &= -U + V_\perp + t_{\mathrm{pair}} + (2J_\perp + 3J_\perp^z)/4.
\end{aligned}$$

This duality relation leads to the conclusion that, if CDW or *d*DW order, for example, appears in a certain parameter region, spin current (*d*SDW) or SDW order exists in a dual region, respectively. Note that even if we start from a spin isotropic model ($J_\perp = J_\perp^z$), the dual model is spin anisotropic and hence the spin symmetry breaking associated with SDW or *d*SDW order can occur in the dual model. The whole phase diagram must be symmetric with respect to the self-dual space (33) and the direct phase transitions between dual phases, if ever, locate exactly on the self-dual space.

4.3 Duality relation between spin and charge

A combination of the duality transformations (15) and (22) leads to the spin-charge duality transformation given by

$$\begin{aligned}
\hat{d}_{j,+,\uparrow} &= d_{j,+,\uparrow}, & \hat{d}_{j,+,\downarrow} &= -id_{j,+,\downarrow},\\
\hat{d}_{j,-,\uparrow} &= d_{j,-,\uparrow}, & \hat{d}_{j,-,\downarrow} &= id_{j,-,\downarrow},
\end{aligned} \qquad (35)$$

which directly exchanges charge and spin degrees of freedom. For example, general density-wave operators $\sum_{\mathbf{k},\sigma} f_A(\mathbf{k}) c_\sigma^\dagger(\mathbf{k}) c_\sigma(\mathbf{k}+\mathbf{Q})$ are transformed to spin-density-wave operators $\sum_{\mathbf{k}} \sum_{\sigma,\sigma'=\uparrow,\downarrow} f_A(\mathbf{k}) c_\sigma^\dagger(\mathbf{k}) \sigma^z_{\sigma\sigma'} c_{\sigma'}(\mathbf{k}+\mathbf{Q})$.

The transformation (35) gives another duality relation in the Hamiltonian, which is given by the combination of eqs. (31) and (34). The model is self-dual in the space satisfying both $U - V_\perp + J_\perp^z/4 = 0$ and $t_{\mathrm{pair}} + J_\perp/2 = 0$, which is the intersection of the self-dual spaces (30) for duality I and (33) for duality II.

4.4 Quantum phase transition at self-dual points

Here we briefly discuss the nature of quantum phase transitions at self-dual points. In the self-dual spaces (30) and (33), the model Hamiltonian is invariant under the continuous rotations given by the unitary operators $U_I(\theta)$ and $U_{II}(\theta)$, respectively. [This can be easily seen in eq. (28) by setting $A = 0$ or $B = 0$.] Hence the self-dual models have extra hidden U(1) symmetries. Because of the U(1) symmetries in the self-dual space, a rigorous theorem[25] concludes that the dual orders that are continuously transformed to each other disappear on the self-dual models in one dimension. Hence, the direct phase transition between the dual phases, if it exists, must be of second order. See Fig. 2. An exception can appear only if susceptibility of the generator for this rotation is diverging at the transition (see ref. 25); In this case, the orders can survive even at the transition point in the self-dual space, but there are gapless excitations associated with the continuous U(1) symmetry breakdown. We note that the discussion above does not exclude the presence of gapful phases in the self-dual models. Self-dual phases, which can be either gapful or gapless, may appear in a finite parameter region including the self-dual space.

In the quantum phase transitions at self-dual points, the A- and B-terms in the interaction (28) serve as symmetry-breaking perturbations for the U(1) symmetries associated with $U_I(\theta)$ and $U_{II}(\theta)$, respectively. Actually, the A- and B-terms can be expressed with the difference of dual operators in the forms

$$\begin{aligned}
&A(d^\dagger_{j,+,\uparrow} d_{j,-,\uparrow} d^\dagger_{j,+,\downarrow} d_{j,-,\downarrow} + \mathrm{H.c.})\\
&= A[(\mathcal{O}_{\mathrm{CDW}})^2 - (\mathcal{O}_{d\mathrm{DW}})^2] = A[(\mathcal{O}_{d\mathrm{SDW}})^2 - (\mathcal{O}_{\mathrm{SDW}})^2]\\
&= A[|\mathcal{O}_{s\mathrm{SC}}|^2 - |\mathcal{O}_{d\mathrm{SC}}|^2], \qquad (36)\\
&B(d^\dagger_{j,+,\uparrow} d_{j,-,\uparrow} d^\dagger_{j,-,\downarrow} d_{j,+,\downarrow} + \mathrm{H.c.})\\
&= B[(\mathcal{O}_{\mathrm{CDW}})^2 - (\mathcal{O}_{d\mathrm{SDW}})^2] = B[(\mathcal{O}_{d\mathrm{DW}})^2 - (\mathcal{O}_{\mathrm{SDW}})^2]\\
&= B[|\mathcal{O}_{s,o}|^2 - |\mathcal{O}_{t,o}|^2], \qquad (37)
\end{aligned}$$

where $|\mathcal{O}|^2 = (\mathrm{Re}\mathcal{O})^2 + (\mathrm{Im}\mathcal{O})^2$. When a symmetry-breaking perturbation is relevant, it induces a order and a gap determining the criticality of the phase transition (Fig. 2). For example, the transition between CDW and *d*DW phases was studied using the Bosonization method in the weak-coupling limit and shown to be characterized by $c = 1$ Gaussian criticality.[10,13]

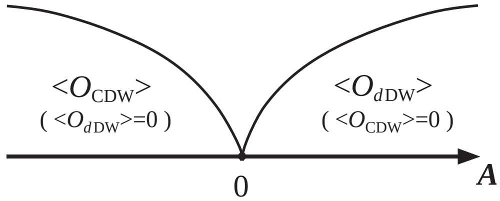

Fig. 2. Orders of dual phases are destabilized at the self-dual points, i.e., $A = 0$ for the duality I and $B = 0$ for the duality II, because of the hidden U(1) symmetries. Hence the direct phase transitions between dual phases are of second order.

5. The SO(5) symmetric ladder

The SO(5) symmetric ladder presented by Scalapino *et al.*[23] has only intra-rung couplings and belongs to the Hamiltonian (2). They showed that the half-filled Hubbard ladder ($\mu = \frac{1}{2}U + V$) has the SO(5) symmetry if the parameters satisfy

$$J_\perp = J_\perp^z = 4(U + V_\perp). \qquad (38)$$

No condition is imposed on the values of $t_\perp$, $t_\parallel$, and t_{pair}, since the hopping terms in H_0 and the pair-hopping term (t_{pair}) are SO(5) symmetric.[23,24] In this section, we show a duality relation in the SO(5) symmetric ladder.

Scalapino *et al.*[23] showed that the five-dimensional SO(5) superspin vector n^a ($a = 1, \ldots, 5$) is related to

the AFM and dSC operators by

$$n^1 = \sqrt{2}\mathrm{Re}\mathcal{O}_{d\mathrm{SC}}, \qquad n^{(2,3,4)} = S_1^{(x,y,z)} - S_2^{(x,y,z)},$$
$$n^5 = \sqrt{2}\mathrm{Im}\mathcal{O}_{d\mathrm{SC}}, \tag{39}$$

where we have omitted rung indices j. In terms of the rung charge $Q = \frac{1}{2}\sum_\sigma (n_{1\sigma} + n_{2\sigma} - 1)$, the rung spin $S^\alpha = S_1^\alpha + S_2^\alpha$, and the π_α operators $\pi_\alpha^\dagger = -\frac{1}{2}\sum_{\sigma,\sigma'} c_{1\sigma}^\dagger (\sigma_\alpha \sigma_y)_{\sigma\sigma'} c_{2\sigma'}^\dagger$, the ten-dimensional SO(5) symmetry generators L^{ab} $(a, b = 1, \dots, 5)$ are expressed with

$$\begin{bmatrix} 0 & & & & \\ \pi_x^\dagger + \pi_x & 0 & & & \\ \pi_y^\dagger + \pi_y & -S^z & 0 & & \\ \pi_z^\dagger + \pi_z & S^y & -S^x & 0 & \\ Q & \frac{1}{i}(\pi_x^\dagger - \pi_x) & \frac{1}{i}(\pi_y^\dagger - \pi_y) & \frac{1}{i}(\pi_z^\dagger - \pi_z) & 0 \end{bmatrix} \tag{40}$$

where the elements are antisymmetric. The SO(5) scalar operator ρ is expressed by the charge density operator with

$$\rho = \mathcal{O}_{\mathrm{CDW}} + 1. \tag{41}$$

Applying the duality transformation I, we find that the transformed superspin vector $\tilde{n}^a$ is related to the spin current (dSDW) and sSC operators by

$$\tilde{n}^1 = \sqrt{2}\mathrm{Re}\mathcal{O}_{s\mathrm{SC}}, \qquad \tilde{n}^5 = \sqrt{2}\mathrm{Im}\mathcal{O}_{s\mathrm{SC}},$$
$$\tilde{n}^{(2,3,4)} = -\frac{i}{2}\sum_{\sigma,\sigma'} \sigma_{\sigma\sigma'}^{(x,y,z)} (c_{1,\sigma}^\dagger c_{2,\sigma'} - \mathrm{H.c.}) = j_s^{(x,y,z)}, \tag{42}$$

where the three elements j_s^α $(\alpha = x, y, z)$ are Cartesian components of the spin current (dSDW) along each rung and $j_s^z = \mathcal{O}_{d\mathrm{SDW}}$. Moreover, using the relations $\tilde{\boldsymbol{S}} = \boldsymbol{S}$, $\tilde{\boldsymbol{\pi}} = i\boldsymbol{\pi}$, and $\tilde{Q} = Q$, one can find that the transformed symmetry generators $\tilde{L}^{ab}$ are given by just a permutation of original generators L^{ab},

$$\tilde{L}^{(2,3,4)1} = L^{5(2,3,4)}, \qquad \tilde{L}^{5(2,3,4)} = -L^{(2,3,4)1}, \tag{43}$$

and the rest of L^{ab} $(a \geq b)$ are unchanged. Thus, the set of SO(5) generators are invariant under the duality transformation I, except for the permutation of elements. From these results, we can conclude that, besides the superspin vector (39), the same SO(5) symmetry with the generators (40) unifies the dSDW and sSC in another grand order parameter field

$$(\sqrt{2}\mathrm{Im}\mathcal{O}_{s\mathrm{SC}}, j_s^x, j_s^y, j_s^z, -\sqrt{2}\mathrm{Re}\mathcal{O}_{s\mathrm{SC}}). \tag{44}$$

The transformed SO(5) scalar operator $\tilde{\rho}$ is given by the dDW operator with

$$\tilde{\rho} = -\mathcal{O}_{d\mathrm{DW}} + 1. \tag{45}$$

One may be surprised that the dDW operator is an SO(5) scalar, but one can easily express the dDW (and CDW) operator in terms of SO(5) spinors in an SO(5) symmetric form.

Since the set of SO(5) generators are invariant under the duality transformation I, the SO(5) symmetric Hubbard ladder is mapped onto the SO(5) symmetric model satisfying the condition (38). One can easily check that the condition (38) is conserved under the parameter mapping (31). However, the parameter point in $(J_\perp, U, t_{pair})$ for the SO(5) symmetric model is converted to the dual point with

$$\tilde{U} = -U - V_\perp - t_{\mathrm{pair}}/2, \qquad \tilde{J}_\perp = 2V_\perp - t_{\mathrm{pair}},$$
$$\tilde{t}_{\mathrm{pair}} = -2U - V_\perp + t_{\mathrm{pair}}/2, \tag{46}$$

and $\tilde{V}_\perp$ is given by $\tilde{V}_\perp = \tilde{J}_\perp/4 - \tilde{U}$.

To demonstrate the duality relation in the SO(5) symmetric model, we write down the Hamiltonian in terms of dual and self-dual operators. The intra-rung part (3b) can be cast into the form

$$H_{\mathrm{rung}} - \mu_0 \sum_{j,l,\sigma} n_{j,l,\sigma} = \sum_j \left[\left(\frac{J_\perp}{4} - \frac{t_{\mathrm{pair}}}{2} \right) \sum_{a<b} (L_j^{ab})^2 + \left(\frac{J_\perp}{2} + 2U - t_{\mathrm{pair}} \right) (\rho_j - 1)^2 - 2t_{\mathrm{pair}} (\tilde{\rho}_j - 1)^2 \right], \tag{47}$$

where $\mu_0 = \frac{1}{2}U + V$. The Casimir operator $C = \sum_{a<b} (L^{ab})^2$ is invariant under the duality transformation, and the kinetic energy term as well. It is also instructive to rewrite the Hamiltonian in terms of the dual superspin vectors in the form

$$H_{\mathrm{rung}} - \mu_0 \sum_{j,l,\sigma} n_{j,l,\sigma} = \sum_j \left[\left(\frac{J_\perp}{20} - \frac{4U}{5} + \frac{7t_{\mathrm{pair}}}{10} \right) \sum_{a<b} (L_j^{ab})^2 - \frac{1}{10} (J_\perp + 4U - 2t_{\mathrm{pair}}) \sum_a (n_j^a)^2 + \frac{2}{5} t_{\mathrm{pair}} \sum_a (\tilde{n}_j^a)^2 \right], \tag{48}$$

where we have used the Fierz identity[23]

$$5(\rho - 1)^2 = 5 - (\boldsymbol{n})^2 - 2\sum_{a<b} (L^{ab})^2. \tag{49}$$

The duality relation in the Hamiltonian is transparent in the forms (47) and (48). The SO(5) symmetric Hamiltonian is self dual if two coefficients of dual operators are equal, i.e.,

$$J_\perp + 4U + 2t_{\mathrm{pair}} = 0. \tag{50}$$

The self-dual SO(5) model has a superspin SO(5) × duality U(1) symmetry, in total. The two-dimensional self-dual plane divides the three-dimensional SO(5) symmetric parameter space into two subspaces. In one subspace, the AFM and dSC correlations are dominant, showing a crossover upon doping as was discussed in refs. 20–23. In the other subspace, the dSDW and sSC correlations come out to be relevant orders and hence symmetry breaking perturbations enhance dSDW or sSC order in the same way as the case of the AFM and dSC orders.

For $t_{\rm pair} = 0$ and $U < 0$, this self-dual plane (50) coincides with a phase boundary line found in the previous studies.[10–12,23,24] For example, the transition line between the CDW and dDW phases given in ref. 11 is identical with the self-dual line. For $t_{\rm pair} = 0$ and $U > 0$, on the other hand, the self-dual space locates in the C2S2 critical phase. This implies that the critical phase is self-dual and a crossover between dual states characterized by different dominant correlations occurs on this self-dual line.

6. Duality relations in weak coupling

Here we reconsider the duality relations in the weak-coupling limit. In this limit, the duality relation I corresponds to an already known relation in Bosonization studies.[10,12,13] We also find another new duality relation which appears only in the low-energy effective theory.

6.1 Bosonization framework

We apply the Abelian bosonization method.[26,27] In a continuum limit, we expand the electron-field operators as

$$\psi_{\lambda\sigma}(x) = e^{ik_{F\lambda}x}\psi_{R\lambda\sigma}(x) + e^{-ik_{F\lambda}x}\psi_{L\lambda\sigma}(x) \qquad (51)$$

for $\lambda = \pm$ and $\sigma = \uparrow, \downarrow$, where $\psi_{R\lambda\sigma}$ ($\psi_{L\lambda\sigma}$) represents the chiral fields for right- (left-) moving electrons in the bonding ($\lambda = +$) and antibonding ($\lambda = -$) bands and $k_{F\lambda}$ are the Fermi wave vectors. As in the usual way, we introduce boson fields $\varphi_{p\lambda\sigma}$ for the chiral fields as

$$\psi_{p\lambda\sigma}(x) = \frac{\eta_{\lambda\sigma}}{\sqrt{2\pi a_0}}\exp[ip\varphi_{p\lambda\sigma}(x)], \qquad (52)$$

for $p = R/L = +/-$, where $\eta_{\lambda\sigma}$ denote the Klein factors and a_0 the lattice constant. Then, we define a new set of boson fields:

$$\phi_{\nu r} = \phi^R_{\nu r} + \phi^L_{\nu r}, \qquad \theta_{\nu r} = \phi^R_{\nu r} - \phi^L_{\nu r}, \qquad (53)$$

with

$$\phi^p_{cr} = \frac{1}{4}\{\varphi_{p+\uparrow} + \varphi_{p+\downarrow} + r(\varphi_{p-\uparrow} + \varphi_{p-\downarrow})\}, \qquad (54a)$$

$$\phi^p_{sr} = \frac{1}{4}\{\varphi_{p+\uparrow} - \varphi_{p+\downarrow} + r(\varphi_{p-\uparrow} - \varphi_{p-\downarrow})\} \qquad (54b)$$

for $\nu = c, s$ and $r = \pm$. The ϕ and θ fields (53) satisfy the commutation relations $[\phi_{\nu r}(x), \theta_{\nu' r'}(x')] = -i\pi\Theta(x'-x)\delta_{\nu\nu'}\delta_{rr'}$ [$\Theta(x)$ is the Heaviside step function] and $[\phi_{\nu r}(x), \phi_{\nu' r'}(x')] = [\theta_{\nu r}(x), \theta_{\nu' r'}(x')] = 0$.

Using these boson fields, one finds that the duality transformation I is expressed as the translation of the θ_{cr} fields by $r\pi/2$ while the duality transformation II is the translation of the θ_{c+} and θ_{s-} fields by $\pi/2$. It was already realized that the former transformation gives a duality relation in the Bosonization method.[10,12,13]

6.2 Duality relation III

Here we restrict our discussion to the half-filled system. Applying gauge transformations to right-moving and left-moving chiral fermion fields independently, we can form another duality transformation

$$\check{\psi}_{R\lambda\sigma} = e^{i\pi/4}\psi_{R\lambda\sigma}, \qquad \check{\psi}_{L\lambda\sigma} = e^{-i\pi/4}\psi_{L\lambda\sigma}, \qquad (55)$$

which gives the following relations between order parameters

$$\check{O}_{\rm CDW}(\pi) \sim -O_{p{\rm DW}}(\pi), \qquad \check{O}_{d{\rm DW}}(\pi) \sim O_{f{\rm DW}}(\pi),$$
$$\check{O}_{\rm SDW}(\pi) \sim -O_{p{\rm SDW}}(\pi), \qquad \check{O}_{d{\rm SDW}}(\pi) \sim O_{f{\rm SDW}}(\pi), \qquad (56)$$

where we have omitted $k_{F\lambda}$-dependent prefactors. We thus find that the transformation (55) exchanges rung-centered orders (s- and d-wave orders) and plaquette-centered orders (p- and f-wave orders). See Fig. 1. The low-energy effective Hamiltonian can be decoupled into self-dual parts and symmetry breaking perturbations, in which the duality relation is visible. The duality relation III thereby gives a parameter mapping as similar to the duality relations I and II. In terms of the bosonic fields, this transformation is given by the translation of the ϕ_{c+} field by $\pi/2$ and one can see that the duality relation (56) was transparent in the results of the bosonization studies.[12,13] We note that this transformation can be written in such a compact form only in the low-energy effective theory.

6.3 Ising duality

Finally we mention the Ising duality transformation,[10] which interchanges the ϕ_{s-} and θ_{s-} fields,

$$\theta_{s-} \to \phi_{s-}, \qquad \phi_{s-} \to \theta_{s-}, \qquad (57)$$

and leaves the rest of the ϕ and θ fields unchanged. This transformation converts a dDW state into a d-Mott state and a CDW state into an s-Mott state. Thus, this is a duality between ordered and disordered phases and qualitatively different from the duality relations I-III.

7. Strong coupling approach at half-filling

In this section, we discuss the stability of ordered states at half-filling using strong-coupling expansions as was discussed in ref. 13. Using the duality transformations I and II, we can easily find various transition lines between ordered (density-wave) and disordered (Mott-insulating) phases.

Let us discuss CDW, dDW, SDW, and dSDW orders. For individual phases, an effective Hamiltonian can be derived on doubly degenerate rung basis $|+\rangle$ and $|-\rangle$ in the form

$$H_{\rm eff} = \sum_j (K\tau^z_j\tau^z_{j+1} + h\tau^x_j), \qquad (58)$$

where τ^α_j ($\alpha = x, y, z$) denotes the Pauli matrices, and $\tau^z_j|\pm\rangle = \pm|\pm\rangle$. This Hamiltonian suggests that doubly degenerate ordered ground states appear in $K > |h|$ and disordered states in $K < |h|$. These transitions are characterized by the Ising criticality ($c = 1/2$).

In the following derivation, we use convenient basis,

$$\begin{aligned} |1\rangle &\equiv c^\dagger_{j,1,\uparrow}c^\dagger_{j,2,\downarrow}|0\rangle, \qquad |2\rangle \equiv c^\dagger_{j,1,\downarrow}c^\dagger_{j,2,\uparrow}|0\rangle, \\ |3\rangle &\equiv c^\dagger_{j,1,\uparrow}c^\dagger_{j,1,\downarrow}|0\rangle, \qquad |4\rangle \equiv c^\dagger_{j,2,\uparrow}c^\dagger_{j,2,\downarrow}|0\rangle. \end{aligned} \tag{59}$$

7.1 CDW–s-Mott

The perturbation expansion for a CDW ordered state can be performed in the strong coupling limit $2V_\perp - U \to \infty$ and $-U - J_\perp/2 - J^z_\perp/4 + V_\perp \to \infty$, where two degenerate ground-state basis per rung are $|+\rangle = |3\rangle$ and $|-\rangle = |4\rangle$. By treating hopping terms $(t_\parallel, t_\perp, t_{\text{pair}})$ as the perturbation, the couplings of the effective Hamiltonian are derived as

$$K = 2{t_\parallel}^2/(2V_\perp - U), \tag{60a}$$

$$h = t_{\text{pair}} - 2{t_\perp}^2/(-U + V_\perp - J_\perp/2 - J^z_\perp/4). \tag{60b}$$

A CDW ordered state appears for $K > |h|$ and an s-Mott state for $K < |h|$.

7.2 dDW–d-Mott

Using the duality transformations, we can derive the effective Hamiltonian for other ordered states. For the dDW phase, the unperturbed ground-state basis are $|\pm\rangle = \frac{1}{2}\{(|1\rangle - |2\rangle) \mp i(|3\rangle - |4\rangle)\}$ in the strong coupling limit $V_\perp + t_{\text{pair}} + J_\perp/2 + J^z_\perp/4 \to \infty$ and $U - V_\perp + 3t_{\text{pair}} + J_\perp/2 + J^z_\perp/4 \to \infty$. The couplings of the effective Hamiltonian are given by

$$K = 8{t_\parallel}^2/(4V_\perp + 4t_{\text{pair}} + 2J_\perp + J^z_\perp), \tag{61a}$$

$$\begin{aligned} h = &(-U + V_\perp + t_{\text{pair}})/2 - (2J_\perp + J^z_\perp)/8 \\ &- 4{t_\perp}^2/(U - V_\perp + 3t_{\text{pair}} + J_\perp/2 + J^z_\perp/4). \end{aligned} \tag{61b}$$

One can say that a dDW ordered state appears for $K > |h|$ and a d-Mott state for $K < |h|$.

7.3 SDW–d-Mott

For the SDW phase, $|+\rangle = |1\rangle$ and $|-\rangle = -|2\rangle$ in the strong coupling limit $2U + J^z_\perp \to \infty$ and $U - V_\perp + t_{\text{pair}} + J^z_\perp/4 \to \infty$. The couplings of the effective Hamiltonian are derived as

$$K = 4{t_\parallel}^2/(J^z_\perp + 2U), \tag{62a}$$

$$h = -J_\perp/2 - 2{t_\perp}^2/(U - V_\perp + t_{\text{pair}} + J^z_\perp/4). \tag{62b}$$

One can say that an SDW ordered state appears for $K > |h|$ and a d-Mott state for $K < |h|$.

7.4 dSDW–s-Mott

For the dSDW phase, $|\pm\rangle = \frac{1}{2}\{(|3\rangle + |4\rangle) \pm i(|1\rangle + |2\rangle)\}$ in the strong coupling limit $V_\perp - t_{\text{pair}} - J_\perp/2 + J^z_\perp/4 \to \infty$ and $-U + V_\perp - t_{\text{pair}} - 3J_\perp/2 - J^z_\perp/4 \to \infty$. The couplings of the effective Hamiltonian are derived as

$$K = 8{t_\parallel}^2/(4V_\perp - 4t_{\text{pair}} - 2J_\perp + J^z_\perp), \tag{63a}$$

$$\begin{aligned} h = &(U - V_\perp + t_{\text{pair}})/2 - (2J_\perp - J^z_\perp)/8 \\ &- 4{t_\perp}^2/(-U + V_\perp - t_{\text{pair}} - 3J_\perp/2 - J^z_\perp/4). \end{aligned} \tag{63b}$$

One can say that a dSDW (spin current) ordered state appears for $K > |h|$ and an s-Mott state for $K < |h|$.

8. Discussions

In summary, we have established duality relations in correlated electron systems on the two-leg ladder. Our arguments clarify mutual relations between conventional and various unconventional density-wave orders, as shown in Fig. 1. The present duality arguments reveal that the stability of unconventional density-wave orders such as staggered flux (dDW) and circulating spin current (dSDW) is equal to that of conventional density-wave orders in the dual parameter spaces. Recently, large-scale numerical analyses[14] reported the appearance of incommensurate dDW (quasi-)long range order upon doping. Applying the duality relations to this result, we can immediately conclude that new incommensurate CDW, SDW, and dSDW phases stably exist in the doped generalized Hubbard ladder in the dual parameter spaces. These duality arguments can be easily generalized to the Hubbard ladder including inter-rung interactions.

Next, we found a duality structure in the SO(5) symmetric Hubbard ladder system. The SO(5) symmetry, which was proposed to unify AFM and dSC, also unifies dSDW and sSC. This gives a new route to s-wave superconductivity in strongly correlated electron systems. If the coupling parameters of a half-filled system is close to the SO(5) symmetric region and the circulating spin current correlation is dominant in the ground state, hole doping can be a symmetry breaking perturbation which enhances the s-wave superconductivity. Thus this system would show a crossover from a dSDW dominant state to a sSC one upon doping.

Finally, we give a remark on the spin-chirality duality which was previously introduced for the spin ladder.[28,29] In the spin-chirality duality transformation, antiferromagnetic spin and vector chirality degrees of freedom are converted to each other. One can see that this spin-chirality duality has an analogy with the duality I for electron systems: in the duality I, SDW (antiferromagnetic spin order) is related to spin current, which is expected to have a spin vector chiral order. Actually, Lecheminant and Totsuka[30] showed in a Majorana fermion representation that the spin-chirality duality transformation can be written as a gauge transformation for the fermions, which is similar to the transformation given in the present paper.

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